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Asymptotic Expressions of Certain Type of Matrix Integrals

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Abstract—This paper deals with the asymptotic expression of parametric matrix integrals related to Laguerre's matrix polynomials using path integration of matrix valued functions. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Consider the definite integral

$$I = I(\lambda) = \int_{-1}^{1} f(mx)\left(1 - x^2\right)^{A-I/2} e^{(1/2)m^2x^2 + i\lambda mx}\, dx, \tag{1.1}$$

where $f(x)$ is a given polynomial, $m > 0$, $\lambda > 0$, and A is a matrix in $\mathbb{C}^{r\times r}$ such that if I is the identify matrix in $\mathbb{C}^{r\times r}$, one satisfies

$$A + nI \text{ is invertible for all integer } n \geq 0 \tag{1.2}$$

and

$$\beta(A) = \min\{\mathrm{Re}(z); z \text{ eigenvalue of } A\} > -\frac{1}{2}. \tag{1.3}$$

In order to find the asymptotic expression of (1.1), we introduce the path of integration which consists of three rectilinear parts joining, respectively, the points: $\gamma_1(t) = -1 + it$, $0 \leq t \leq T$; $\gamma_2(t) = t + iT$, $-1 \leq t \leq 1$; $\gamma_3(t) = 1 + it$, $0 \leq t \leq T$, then the residue theorem is applied. If

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B is a matrix in $\mathbb{C}^{r\times r}$ and $\|B\| = \max\{\sqrt{z}; z$ eigenvalue of $B^H B\}$, where B^H is the conjugate transpose of B, then by [1, p. 556] it follows that

$$\left\|e^{tB}\right\| \le e^{t\alpha(B)} \sum_{k=0}^{r-1} \frac{\left(\|B\| r^{1/2} t\right)^k}{k!}, \qquad t \ge 0, \tag{1.4}$$

where $\alpha(B) = \max\{\mathrm{Re}(z); z$ eigenvalue of $B\}$. By (1.4), using the notation of (1.3), one gets

$$\begin{aligned} \left\|t^A\right\| = \left\|e^{A\ln t}\right\| = \left\|e^{-A(-\ln t)}\right\| &\le e^{\beta(A)(-\ln t)} \sum_{k=0}^{r-1} \frac{\left(-\|A\| r^{1/2}\ln t\right)^k}{k!}, \qquad 0 < t < 1, \\ \left\|t^A\right\| &\le e^{\alpha(A)\ln t} \sum_{k=0}^{r-1} \frac{\left(\|A\| r^{1/2}\ln t\right)^k}{k!}, \qquad t \ge 1. \end{aligned} \tag{1.5}$$

If z is a complex number with positive real part, $|z| \ge 1$, and $\ln z$ denotes the principal logarithm [2, p. 72], then

$$\left\|z^A\right\| = \left\|e^{A\ln z}\right\| = \left\|e^{A(\ln|z| + i\arg z)}\right\| \le e^{\|A\|\,|\ln|z||} e^{\pi\|A\|} = |z|^{\|A\|} e^{\pi\|A\|}. \tag{1.6}$$

2. THE INTEGRAL AS A SUM OF TWO IMPROPER INTEGRALS

Consider the matrix valued function F of complex variable z defined by

$$F(z) = f(mz)\left(1 - z^2\right)^{A-I/2} e^{(1/2)m^2z^2 + i\lambda mz}, \tag{2.1}$$

and note that $F(z)$ is holomorphic in the interior of the rectangle $-1 \le \mathrm{Re}(z) \le 1$; $0 \le \mathrm{Im}(z) \le T$; $T > 0$.

In accordance with the notation of the introduction, let $I_2 = \int_{\gamma_2} F$ and note that by (1.6), one gets

$$\begin{aligned} \|I_2\| &\le \int_{-1}^{1} \|f(m(t+iT))\| \left\|\left(1 - (t+iT)^2\right)^{A-I/2}\right\| \left|e^{(1/2)m^2(t+iT)^2}\right| \left|e^{i\lambda m(t+iT)}\right| dt \\ &\le e^{\pi\|A-I/2\|} e^{-(1/2)m^2T^2 - \lambda mT} \int_{-1}^{1} \|f(m(t+iT))\| \left|\left(1 - \left(t^2 - T^2 + 2itT\right)\right)\right|^{\|A-I/2\|} e^{(1/2)m^2t^2}\, dt \\ &\le e^{\pi\|A-I/2\|} e^{-(1/2)m^2T^2 - \lambda mT} e^{(1/2)m^2} \left(\left(1+T^2\right)^2 + 4T^2\right)^{\|A-I/2\|/2} \int_{-1}^{1} \|f(m(t+iT))\|\, dt. \end{aligned}$$

Hence, as $f(mz)$ is a polynomial, it follows that

$$\lim_{T\longrightarrow\infty} I_2(\lambda) = 0, \quad \text{uniformly for } \lambda \to \infty, \quad 0 < m < b, \tag{2.2}$$

where b is a fixed positive number.

If $I_3 = \int_{\gamma_3} F$, it is easy to show that

$$\begin{aligned} \lim_{T\longrightarrow\infty} I_3 = -&\left(\int_0^\infty e^{m^2(it - t^2/2)} e^{-\lambda mt} f(m(1+it))\, t^{A-I/2} \left(1 + \frac{it}{2}\right)^{A-I/2} dt\right) \\ &\times 2^{A-I/2} e^{-i((2A+I)/4)\pi} e^{(1/2)m^2 + i\lambda m}. \end{aligned} \tag{2.3}$$

If $I_1 = \int_{\gamma_1} F$, in an analogous way one gets

$$\begin{aligned} \lim_{T\longrightarrow\infty} I_1 = &\left(\int_0^\infty e^{-m^2(it + t^2/2)} e^{-\lambda mt} f(m(-1+it))\, t^{A-I/2} \left(1 - \frac{it}{2}\right)^{A-I/2} dt\right) \\ &\times 2^{A-I/2} e^{i((2A+I)/4)\pi} e^{(1/2)m^2 - i\lambda m}. \end{aligned} \tag{2.4}$$

Considering the substitution $u = \lambda m t$ in (2.3),(2.4), it follows that

$$\lim_{T\longrightarrow\infty} I_3 = \left(\int_0^\infty e^{im(u/\lambda)-(1/2)(u/\lambda)^2} e^{-u} f\left(m + \frac{i}{u}\lambda\right) u^{A-I/2} \left(1 + \frac{iu}{2\lambda m}\right)^{A-I/2} du\right) \times 2^{A-I/2}(\lambda m)^{-A-I/2} e^{(1/2)m^2+i\lambda m} e^{-i((2A+I)/4)\pi}, \tag{2.5}$$

$$\lim_{T\longrightarrow\infty} I_1 = \left(\int_0^\infty e^{-im(u/\lambda)-(1/2)(u/\lambda)^2} e^{-u} f\left(-m + \frac{iu}{\lambda}\right) u^{A-I/2} \left(1 - \frac{iu}{2\lambda m}\right)^{A-I/2} du\right) \times 2^{A-I/2}(\lambda m)^{-A-I/2} e^{(1/2)m^2-i\lambda m} e^{i((2A+I)/4)\pi}. \tag{2.6}$$

We conclude this section considering the asymptotic expression of (1.1) for the case where $f(x) = 1$. Note that by (1.5), (1.6), and (2.5), if $f(x) = 1$, it follows that

$$\begin{aligned}
\lim_{T\longrightarrow\infty} \|I_3\| &\le \left\|2^{A-I/2}(\lambda m)^{-A-I/2} e^{(1/2)m^2}\right\| \left\|e^{-i((2A+I)/4)\pi}\right\| \int_0^\infty e^{-u} \left\|u^{A-I/2}\right\| \\
&\times \left\|\left(1 + \frac{iu}{2\lambda m}\right)^{A-I/2}\right\| du \le \left\|2^{A-I/2}(\lambda m)^{-A-I/2} e^{(1/2)m^2}\right\| \left\|e^{-i((2A+I)/4)\pi}\right\| \\
&\times e^{(\pi/2)\|A-I/2\|} \left\{\sum_{j=0}^{r-1}\sum_{k=0}^{r-1} \frac{\left(-\|A-I/2\|\sqrt{r}\right)^j}{j!} \frac{\left(\|A-I/2\|\sqrt{r}\right)^k}{k!} \times \int_0^1 e^{-u}\left(\ln^j u\right)\right. \\
&\times u^{\beta(A)-1/2}\left(1 + \frac{u^2}{4\lambda^2 m^2}\right)^{(\alpha(A))/2-1/4} \left(\ln^k \sqrt{1 + \left(\frac{u}{2\lambda m}\right)^2}\right) du \\
&+ \sum_{j=0}^{r-1}\sum_{k=0}^{r-1} \frac{\left(\|A-I/2\|\sqrt{r}\right)^j}{j!} \times \frac{\left(\|A-I/2\|\sqrt{r}\right)^k}{k!} \int_1^\infty e^{-u}\left(\ln^j u\right) u^{\alpha(A)-1/2} \\
&\left.\times \left(1 + \frac{u^2}{4\lambda^2 m^2}\right)^{(\alpha(A))/2-1/4} \left(\ln^k \sqrt{1 + \left(\frac{u}{2\lambda m}\right)^2}\right) du\right\}.
\end{aligned} \tag{2.7}$$

Taking into account

$$\begin{aligned}
&\left[1 + \left(\frac{u}{2\lambda m}\right)^2\right]^{(\alpha(A))/2-1/4} \le \left(1 + u^2\right)^{(\alpha(A))/2-1/4}, && \text{if } \alpha(A) \ge \frac{1}{2}, \quad u \ge 1, \\
&\left[1 + \left(\frac{u}{2\lambda m}\right)^2\right]^{(\alpha(A))/2-1/4} \le \left(1 + \frac{1}{4}\right)^{(\alpha(A))/2-1/4}, && \text{if } \alpha(A) \ge \frac{1}{2}, \quad 0 < u < 1, \\
&\left[1 + \left(\frac{u}{2\lambda m}\right)^2\right]^{(\alpha(A))/2-1/4} < 1, && \text{if } \alpha(A) < \frac{1}{2}, \quad 0 < u < \infty,
\end{aligned}$$

$$\ln^k \sqrt{1 + \left(\frac{u}{2\lambda m}\right)^2} \le \ln^k \sqrt{1 + \left(\frac{u}{2}\right)^2},$$

$$\int_0^1 x^m \ln^n x \, dx = \frac{(-1)^n n!}{(m+1)^{m+1}}, \qquad m > -1, \quad n = 0, 1, 2, \ldots,$$

and (2.7) one gets

$$\limsup_{T\longrightarrow\infty} \|I_3\| \le \left\|2^{A-I/2}(\lambda m)^{-A-I/2} e^{(1/2)m^2}\right\| \left\|e^{-i((2A+I)/4)\pi}\right\| e^{(\pi/2)\|A-I/2\|} Q, \tag{2.8}$$

where Q is a positive number independent of λ and $\lambda m \ge 1$.

In an analogous way, by (2.6), one gets

$$\limsup_{T\longrightarrow\infty} \|I_1\| \leq \left\|2^{A-I/2}(\lambda m)^{-A-I/2}e^{(1/2)m^2}\right\| \left\|e^{i((2A+I)/4)\pi}\right\| e^{(\pi/2)\|A-I/2\|}Q. \tag{2.9}$$

By the residue theorem [3, p. 241], (1.5), (2.2), (2.8), and (2.9), it follows that

$$\begin{gathered}\lim_{T\longrightarrow\infty} \|I(\lambda)\| \leq M(\lambda m)^{-\beta(A)+1/2}\sum_{k=0}^{r-1}\frac{\left(\|A\|\sqrt{r}\ln(\lambda m)\right)^k}{k!},\\ \beta(A) > -\frac{1}{2}, \qquad f(x)=1, \qquad \lambda m \geq 1,\end{gathered} \tag{2.10}$$

for some positive constant M.

3. ASYMPTOTIC EXPRESSION: THE GENERAL CASE

Consider the matrix function

$$\varphi(u) = e^{im(u/\lambda)-(1/2)(u/\lambda)^2} f\left(m+\frac{iu}{\lambda}\right)\left(1+\frac{iu}{2\lambda m}\right)^{A-I/2},$$

and note that

$$\begin{aligned}
\varphi(0) &= If(m),\\
\varphi'(0) &= \frac{i\left[mf'(m)+\left(m^2I+(2A-I)/4\right)f(m)\right]}{\lambda m},\\
\varphi'(u) &= \left[\left(\frac{im}{\lambda}-\frac{u}{\lambda^2}\right)f\left(m+\frac{iu}{\lambda}\right)+f'\left(m+\frac{iu}{\lambda}\right)\frac{i}{\lambda}+\frac{if\left(m+(iu)/\lambda\right)(A-I/2)}{2\lambda m\left(1+(iu/2\lambda m)\right)}\right]\\
&\quad\times e^{im(u/\lambda)-(1/2)(u/\lambda)^2}\left(1+\frac{iu}{2\lambda m}\right)^{A-I/2},\\
\varphi''(u) &= \left[-\frac{1}{\lambda^2}f\left(m+\frac{iu}{\lambda}\right)+\left(\frac{im}{\lambda}-\frac{u}{\lambda^2}\right)\frac{i}{\lambda}f'\left(m+\frac{iu}{\lambda}\right)+\left(\frac{i}{\lambda}\right)^2 f''\left(m+\frac{iu}{\lambda}\right)+\left(\frac{i}{\lambda}f'\left(m+\frac{iu}{\lambda}\right)\right.\right.\\
&\quad\times\left.\left.\left(1+\frac{iu}{2\lambda m}\right)^{-1}-f\left(m+\frac{iu}{\lambda}\right)\left(1+\frac{iu}{2\lambda m}\right)^{-2}\frac{i}{2\lambda m}\right)\frac{i(A-I/2)}{2\lambda m}\right]e^{imu/\lambda-(1/2)(u/\lambda)^2}\\
&\quad\times\left(1+\frac{iu}{2\lambda m}\right)^{A-I/2}+\left[\left(\frac{im}{\lambda}-\frac{u}{\lambda^2}\right)f\left(m+\frac{iu}{\lambda}\right)+f'\left(m+\frac{iu}{\lambda}\right)\frac{i}{\lambda}\right.\\
&\quad\left.+\frac{if\left(m+iu/\lambda\right)(A-I/2)}{2\lambda m\left(1+(iu/2\lambda m)\right)}\right]\times\left[\left(\frac{im}{\lambda}-\frac{u}{\lambda^2}\right)+\frac{i}{2\lambda m}\left(A-\frac{I}{2}\right)\left(1+\frac{iu}{2\lambda m}\right)^{-1}\right]\\
&\quad\times e^{im(u/\lambda)-(1/2)(u/\lambda)^2}\left(1+\frac{iu}{2\lambda m}\right)^{A-I/2}=\left[W_1(u)+W_2(u)\left(A-\frac{I}{2}\right)+W_3(u)\left(A-\frac{I}{2}\right)^2\right]\\
&\quad\times\frac{1}{\lambda^2}e^{im(u/\lambda)-(1/2)(u/\lambda)^2}\left(1+\frac{iu}{2\lambda m}\right)^{A-I/2}=\frac{W(u)}{\lambda^2},
\end{aligned} \tag{3.1}$$

where $W_i(u)$ are scalar functions of scalar variable u, whose absolute value increases less rapidly than a certain power of u and which remain bounded for large values of λ if $0 < a \leq m \leq b$.

By (1.6), it follows that $\|(1+(iu/2\lambda m))^{A-I/2}\| \leq (1+(u/(2\lambda m))^2)^{\|A-I/2\|/2}e^{(\pi/2)\|A-I/2\|}$, and if $W(u) = (w_{ij}(u))_{1\leq i,j\leq r}$, one gets that $|w_{ij}(u)|$ remains bounded for large values of u and λ, if $0 < a \leq m \leq b$. Applying Taylor's development, one gets

$$\varphi(u) = \varphi(0)+\varphi'(0)u+\frac{u^2}{2\lambda^2}(w_{ij}(\xi_{ij}(u))), \qquad \xi_{ij}(u)\in]0,u[, \quad 1\leq i,j\leq r, \tag{3.2}$$

where

$$W(u,A,\lambda,m) = (w_{ij}(\xi_{ij}(u)),$$

and

$$\|W(u,A,\lambda,m)\| < W, \qquad \text{if } \lambda \to \infty, \quad u \to \infty, \quad 0 < a \le m \le b.$$

Substituting (3.2) in expression (2.3) of I_3, it follows that

$$I_3 = \left\{ \int_0^\infty e^{-u} \left[\varphi(0) + \varphi'(0)u + \frac{u^2}{2\lambda^2} W(u,A,\lambda,m) \right] u^{A-I/2}\, du \right\} \times 2^{A-I/2} (\lambda m)^{-A-I/2} e^{(1/2)m^2 + i\lambda m} e^{-i((2A+I)/4)\pi}.$$

By (1.2), one gets (see [4])

$$\int_0^\infty e^{-u} u^{A-I/2} u^n\, du = \Gamma\left(A + \left(n + \frac{1}{2}\right) I\right), \qquad n \ge 0,$$

and

$$\begin{aligned} I_3 = &\left[\varphi(0) + \varphi'(0)\left(A + \frac{I}{2}\right) + \frac{R_1}{\lambda^2}\right] \\ &\times 2^{A-I/2} (\lambda m)^{-A-I/2} e^{(1/2)m^2 + i\lambda m} e^{-i((2A+I)/4)\pi} \Gamma\left(A + \frac{1}{2} I\right), \end{aligned} \tag{3.3}$$

where

$$R_1 = \frac{1}{2}\left[\int_0^\infty e^{-u} W(u,A,\lambda,m) u^{A+(5/2)I}\, du\right] \Gamma^{-1}\left(A + \frac{1}{2} I\right).$$

By (1.5), one gets

$$\begin{aligned} \|R_1\| &\le \frac{1}{2} W \left\| \Gamma^{-1}\left(A + \frac{1}{2} I\right)\right\| \int_0^\infty e^{-u} \left\| u^{A+(5/2)I} \right\| du \\ &\le M \sum_{k=0}^{r-1} \frac{\left(\|A+(5/2)I\|\, r^{1/2}\right)^k}{k!} \left(\int_0^1 e^{-u} u^{\beta(A)+5/2} (-\ln u)^k\, du + \int_1^\infty e^{-u} u^{\alpha(A)+5/2} (\ln u)^k\, du \right). \end{aligned}$$

Since by (1.3) $\alpha(A) \ge \beta(A) > -1/2$, it follows that R_1 is bounded as $\lambda \to \infty$ if $0 < a \le m \le b$. In an analogous way, considering I_1 and

$$\varphi_1(u) = e^{-im(u/\lambda) - (1/2)(u/\lambda)^2} f\left(-m + \frac{iu}{\lambda}\right) \left(1 - \frac{iu}{2\lambda m}\right)^{A-I/2} = \varphi(u,-m),$$

it follows that

$$\begin{aligned} I_1 = &\left[\varphi(0,-m) + \varphi'(0,-m)\left(A + \frac{I}{2}\right) + \frac{R_2}{\lambda^2}\right] \\ &\times 2^{A-I/2} (\lambda m)^{-A-I/2} e^{(1/2)m^2 - i\lambda m} e^{i((2A+I)/4)\pi} \Gamma\left(A + \frac{1}{2} I\right), \end{aligned} \tag{3.4}$$

where R_2 satisfies the same properties as R_1. By the residue theorem, (3.3) and (3.4), one gets

$$\begin{aligned} I(\lambda) = \lim_{T \to \infty} (I_1 + I_3) = &\left[\left(f(m) + i\frac{G(m)}{\lambda} + \frac{R_1}{\lambda^2} \right) \frac{e^{i(\lambda m I - ((2A+I)/4)\pi)}}{2} \right. \\ &\left. + \left(f(-m) + i\frac{G(-m)}{\lambda} + \frac{R_2}{\lambda^2} \right) \frac{e^{-i(\lambda m I - ((2A+I)/4)\pi)}}{2} \right] 2^{A+I/2} (\lambda m)^{-A-I/2} e^{(1/2)m^2} \Gamma\left(A + \frac{1}{2} I\right), \end{aligned} \tag{3.5}$$

where

$$G(m) = \frac{\left[m f'(m) + \left(m^2 I + (2A - I)/4\right) f(m) \right]}{m} \left(A + \frac{I}{2}\right).$$

Finally, by (3.5), one gets the asymptotic expression

$$
\begin{aligned}
I &= \int_0^{\pi} e^{(1/2)m^2\cos^2\varphi} f(m\cos\varphi)(\sin\varphi)^{2A} e^{i\lambda m\cos\varphi}\, d\varphi \\
&= \left[\frac{f(m)+f(-m)}{2}\cos\left(\lambda m I - \frac{2A+I}{4}\pi\right) + i\frac{f(m)-f(-m)}{2}\sin\left(\lambda m I - \frac{2A+I}{4}\pi\right)\right. \\
&\quad \left. + i\frac{G(m)+G(-m)}{2\lambda}\cos\left(\lambda m I - \frac{2A+I}{4}\pi\right) + \frac{G(-m)-G(m)}{2\lambda}\sin\left(\lambda m I - \frac{2A+I}{4}\pi\right) + \frac{R}{\lambda^2}\right] \\
&\quad \times 2^{A+I/2}(\lambda m)^{-A-I/2} e^{(1/2)m^2}\Gamma\left(A+\frac{1}{2}I\right),
\end{aligned}
\tag{3.6}
$$

where

$$
R = \frac{1}{2}\left[e^{i(\lambda m I - ((2A+I)/4)\pi)} R_1 + e^{-i(\lambda m I - ((2A+I)/4)\pi)} R_2\right]
$$

remains bounded as $\lambda \to \infty$, if a and b are fixed numbers with $0 < a \le m \le b$.

REFERENCES

1. G. Golub and C.F. Van Loan, *Matrix Computations*, Johns Hopkins Univ. Press, Baltimore, MD, (1989).
2. S. Saks and A. Zygmund, *Analytic Functions*, Elsevier, Amsterdam, (1971).
3. J. Diendonné, *Foundations of Modern Analysis*, Academic Press, New York, (1960).
4. L. Jódar and J.C. Cortés, Some properties of Gamma and Beta matrix functions, *Appl. Math. Lett.* **11** (1), 89–93, (1998).
5. J.V. Uspensky, On the development of arbitrary functions in series of Hermite's and Laguerre's polynomials, *Annals of Maths.* **28** (2), 593–619, (1927).